JOURNAL OF APPROXIMATION THEORY 71, 239–251 (1992)

Asymptotics for Zeros of Szegő Polynomials Associated with Trigonometric Polynomial Signals

K. PAN

Department of Mathematics, University of California, Riverside, California 92521, U.S.A.

AND

E. B. SAFF*

Institute for Constructive Mathematics, Department of Mathematics, University of South Florida, Tampa, Florida 33620, U.S.A.

Communicated by Paul Nevai

Received January 30, 1992; accepted in revised form March 11, 1992

The study of Wiener–Levinson digital filters leads to certain classes of polynomials orthogonal on the unit circle (Szegő polynomials). Here we present theorems that show that the unknown frequencies in a periodic discrete time signal can be determined from the limiting behavior (as $N \to \infty$) of the zeros of fixed degree Szegő polynomials that are orthogonal with respect to a distribution defined from N successive samples of the signal. This proves an essential part of a conjecture due to Jones, Njåstad, and Saff concerning the frequency analysis problem.

1. INTRODUCTION

We use the term *signal* to denote a doubly-infinite sequence $x = \{x(m)\}_{-\infty}^{\infty}$ of real numbers. Here we are concerned with periodic signals of the form

$$x(m) = \sum_{j=-I}^{I} \alpha_j e^{i\omega_j m}, \qquad x(0) \neq 0, \tag{1.1}$$

where I is a positive integer, the frequencies ω_j satisfy $\omega_j \in \mathbf{R}$, $\omega_{-j} = -\omega_j$,

* Research supported in part by NSF Grant DMS 88-14026.

0021-9045/92 $5.00

$0=\omega_0<\omega_1<\cdots<\omega_I<\pi$ and the coefficients α_j satisfy $\alpha_j\in\mathbf{C}, \alpha_{-j}=\bar{\alpha}_j$, $-I\leqslant j\leqslant I$. In (1.1) we further assume that $\alpha_j\neq 0$, for $j=1,\ldots,I$, and that $\alpha_0\geqslant 0$ (note that we allow the possibility that $\alpha_0=0$). The classical *frequency analysis problem* concerns determining $\omega_1, \omega_2, \ldots, \omega_I$ from N successive samples of the signal (1.1); that is, from the N-truncated causal signal x_N defined by

$$x_N(m):=\begin{cases}\sum_{j=-I}^{I}\alpha_j e^{i\omega_j m}, & 0\leqslant m\leqslant N-1\\ 0, & \text{otherwise.}\end{cases} \tag{1.2}$$

Here we consider a method for solving this problem that is based upon the techniques of Wiener [W] and Levinson [L] and developed in the work of Jones, Njåstad, and Saff [JNS]. The starting point for this method is the *autocorrelation coefficients*

$$\mu_k^{(N)}:=\sum_{m=0}^{N-1} x_N(m)\, x_N(m+k), \qquad k=0,\ \pm 1,\ \pm 2,\ldots. \tag{1.3}$$

As is easily verified (cf. [JNS]), these quantities are the moments of the distribution $d\psi_N(\theta)$, $-\pi\leqslant\theta\leqslant\pi$, defined by

$$d\psi_N(\theta):=|X_N(e^{i\theta})|^2\, d\theta, \tag{1.4}$$

where

$$X_N(z):=\sum_{m=0}^{N-1} x_N(m)\, z^{-m} \tag{1.5}$$

is the Z-transform of the signal x_N. That is,

$$\mu_k^{(N)}=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\theta}\, d\psi_N(\theta), \qquad k=0,\ \pm 1,\ \pm 2,\ldots. \tag{1.6}$$

For $n=0, 1, 2,\ldots$, let

$$\phi_{N,n}(z)=\kappa_{N,n}z^n+\cdots, \qquad \kappa_{N,n}>0, \tag{1.7}$$

denote the unique sequence of orthonormal polynomials on the unit circle (Szegő polynomials) with respect to the distribution function $d\psi_N$. The corresponding monic orthogonal polynomials

$$\Phi_{N,n}(z):=\phi_{N,n}(z)/\kappa_{N,n} \tag{1.8}$$

can be generated from the moments (1.3) by using, for example, Levinson's algorithm [L]. As is well known, the polynomial $\Phi_{N,n}$ satisfies

$$\frac{1}{\kappa_{N,n}^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Phi_{N,n}(z)|^2 \, d\psi_N(\theta)$$
$$= \min_{z^n + \cdots \in \mathscr{P}_n} \frac{1}{2\pi}\int_{-\pi}^{\pi} |z^n + \cdots|^2 \, d\psi_N(\theta), \qquad z = e^{i\theta}, \tag{1.9}$$

where the minimum is taken over all monic polynomials of degree n. Furthermore, all the zeros of $\Phi_{N,n}$ lie in the open unit disk $|z| < 1$ (cf. [Sz, p. 292]).

The main idea of [JNS] is that the zeros of the Szegő polynomials $\Phi_{N,n}$ can be used to determine (approximately) the frequencies ω_j; that is, for suitable N and n, these polynomials should have zeros near the points $e^{i\omega_j}$. A crude motivational argument for this behavior is the following. As shown in [JNS],

$$\frac{1}{N} d\psi_N \xrightarrow{*} \sum_{j=-I}^{I} |\alpha_j|^2 \, \delta_{e^{i\omega_j}} \qquad \text{as} \quad N \to \infty, \tag{1.10}$$

where δ_z denotes the unit point mass at z, and the convergence is with respect to the weak-star topology of measures.[1] Thus for N large, $d\psi_N$ is large near each point $e^{i\omega_j}$ and is relatively small elsewhere on the unit circle. Now in order to achieve the minimization property in (1.9), the polynomial $\Phi_{N,n}(z)$ should be "small" near points where $d\psi_N$ is relatively large; so it is reasonable to expect that $\Phi_{N,n}(z)$ has zeros that are close to the points $e^{i\omega_j}$.

Indeed, Jones, Njåstad, and Saff have made the following

Conjecture [JNS]. If $\alpha_0 > 0$, then as $n + N \to \infty (n \geqslant 2I+1)$ in a suitable manner, the $2I+1$ zeros of $\Phi_{N,n}(z)$ of *largest* modulus approach the points $e^{i\omega_j}$, $-I \leqslant j \leqslant I$. (In case $\alpha_0 = 0$, this conjecture should be modified to refer to the $2I$ zeros of $\Phi_{N,n}(z)$ of largest modulus.)

Some theorems and numerical experiments that support the conjecture are given in [JNS, JS, P]. As one of the main results of the present paper (cf., Theorem 2.4), we prove that the above conjecture is true for any fixed $n \geqslant 2I+1$ and $N \to \infty$. Moreover, for $n = 2I+1$, we show that the rate of this convergence is $O(1/N)$.

Before stating our main results we introduce some further notation. We set

$$W_N(z) := z^{N-1} X_N(z) = \sum_{j=-I}^{I} \alpha_j \frac{z^N - e^{i\omega_j N}}{z - e^{i\omega_j}}, \tag{1.11}$$

[1] In (1.10), $d\psi_N$ is regarded as a measure on the unit circle rather than on $[-\pi, \pi]$.

and let $D_N(z)$ denote the *Szegő function* $D_N(z)$ for the weight $|X_N(e^{i\theta})|^2$:

$$D_N(z) := \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} \log |X_N(e^{i\theta})|^2 \, d\theta\right\}.$$

It is readily seen that $\psi'_N(\theta) = |X_N(e^{i\theta})|^2$ satisfies the *Szegő condition*

$$\int_{-\pi}^{\pi} \psi'_N(\theta)\, d\theta < \infty, \qquad \text{and} \qquad \int_{-\pi}^{\pi} \log \psi'_N(\theta)\, d\theta > -\infty.$$

We also know that (cf. [JNS])

$$D_N(z) = \pm x(0) \prod_{|z_{N,k}| \geqslant 1} (z - z_{N,k}) \prod_{|z_{N,k}| < 1} (1 - \bar{z}_{N,k} z), \tag{1.12}$$

where the $z_{N,k}$'s denote the $(N-1)$ zeros of the polynomial $W_N(z)$ and the sign $(\pm)$ is chosen so that $D_N(0) > 0$.

Finally, for any polynomial p of degree n we denote the *reverse polynomial* of p by

$$p^*(z) := z^n \overline{p(1/\bar{z})}.$$

The outline of the paper is as follows. In Section 2 we state our main results. The proofs of these results are given in Section 3. In Section 4 we present a related theorem that is valid uniformly in n.

2. Statement of Main Results

In this section we state and discuss our main results. Their proofs are given in Section 3. In addition to the notation of the preceding section, we set

$$\beta_j := e^{i\omega_j}, \qquad j = -I, ..., I.$$

THEOREM 2.1. *Assume that* $\alpha_0 > 0$ *in the signal* (1.1). *Then, for each fixed n,* $1 \leqslant n \leqslant 2I+1$,

$$\lim_{N\to\infty} \Phi_{N,n}(z) = \Phi_n(z), \qquad z \in \mathbf{C}, \tag{2.1}$$

where $\Phi_n(z)$ *is the unique monic polynomial of degree n orthogonal with respect to the discrete measure*

$$d\psi := \sum_{j=-I}^{I} |\alpha_j|^2 \, \delta_{\beta_j}. \tag{2.2}$$

In particular,

$$\lim_{N \to \infty} \Phi_{N,2I+1}(z) = \Phi_{2I+1}(z) = \prod_{j=-I}^{I} (z - \beta_j), \qquad z \in \mathbf{C}. \tag{2.3}$$

In (2.1) *and* (2.3), *the convergence is uniform on compact subsets of* **C**. *More precisely, we have for each fixed n,* $1 \leqslant n \leqslant 2I+1$, *and each compact set* $K \subset \mathbf{C}$,

$$|\Phi_{N,n}(z) - \Phi_n(z)| \leqslant A/N, \qquad z \in K, \qquad N = 1, 2, ..., \tag{2.4}$$

where A is a constant that depends on K.

We remark that here and below analogous results hold for the case when $\alpha_0 = 0$; their statements are left to the reader.

Since the measure $d\psi$ in (2.2) is supported in $2I+1$ points on the unit circle, it is easy to see from standard arguments (cf. [Sa]) that all the zeros of $\Phi_1, \Phi_2, ..., \Phi_{2I}$ lie in the open unit disk. Hence, from (2.1) and Hurwitz's theorem we get the following:

COROLLARY 2.2. *If* $\alpha_0 > 0$ *and* $1 \leqslant n \leqslant 2I$, *then, as* $N \to \infty$, *the n zeros of* $\Phi_{N,n}$ *approach n (not necessarily distinct) points in the open unit disk* $|z| < 1$, *namely, the n zeros of* Φ_n.

Of course from (2.3) we can also conclude that the $2I+1$ zeros of $\Phi_{N,2I+1}(z)$ approach the points β_j, $-I \leqslant j \leqslant I$. More precisely, we have

COROLLARY 2.3. *Assume* $\alpha_0 > 0$. *For each N large, let* $\beta_{N,j}$ *denote the zero of* $\Phi_{N,2I+1}(z)$ *that is closest to* β_j. *Then, for* $j = -I, ..., I$,

$$|\beta_{N,j} - \beta_j| = O(1/N), \qquad \text{as } N \to \infty. \tag{2.5}$$

Concerning the frequency analysis problem, Corollaries 2.2 and 2.3 imply that if the degree n of the Szegő polynomials $\Phi_{N,n}$ is strictly less than the number of critical frequency points $e^{i\omega_j}$, then all the zeros of these Szegő polynomials stay away from the unit circle and hence do not converge to any of the points $e^{i\omega_j}$. On the other hand, if the degree of these Szegő polynomials precisely matches the number of points $e^{i\omega_j}$, then their zeros converge to all the points $e^{i\omega_j}$.

Remark. The rate of convergence in (2.5) is, in general, the best possible. This can be seen from the following example for the case when $\alpha_0 = 0$.

Let

$$x(m) := e^{-i\pi m/2} + e^{i\pi m/2} = 2\cos(\pi m/2),$$

so that $\beta_{-1}=-i$, $\beta_1=i$, and $\Phi_2(z)=z^2+1$. On computing the moments $\mu_k^{(N)}$ and using the determinant representation for the orthogonal polynomial $\Phi_{N,2}$, we find

$$\Phi_{N,2}(z)=\begin{cases} z^2+1-2/N, & \text{for } N \text{ even} \\ z^2+1-2/(N+1), & \text{for } N \text{ odd}. \end{cases} \tag{2.6}$$

Thus, the zeros of $\Phi_{N,2}$ approach $\pm i$ with exact rate $1/N$.

We conclude this section with the statement that the conjecture of Jones, Njåstad, and Saff mentioned in the Introduction is true for every fixed $n \geqslant 2I+1$.

THEOREM 2.4. *Assume $\alpha_0>0$. Then for each fixed $n \geqslant 2I+1$, the $2I+1$ zeros of $\Phi_{N,n}(z)$ of largest modulus approach the points β_j, $-I \leqslant j \leqslant I$, as $N \to \infty$.*

Remark. Unlike the case $n \leqslant 2I+1$, the sequence $\{\Phi_{N,n}\}_{N=1}^{\infty}$ for fixed $n>2I+1$ need not have a unique limit. In the proof of Theorem 2.4 we actually establish that every such limit polynomial must be of the form $\Phi_{2I+1}Q$, where Q has all its zeros in the *open* unit disk. Thus for fixed $n>2I+1$, precisely $2I+1$ zeros of $\Phi_{N,n}$ approach the critical frequency points $e^{i\omega_j}$ on the unit circle, while the remaining $n-(2I+1)$ zeros stay away from the unit circle.

In case $\alpha_0=0$, Theorem 2.4 should be modified to read that for fixed $n \geqslant 2I$, the $2I$ zeros of $\Phi_{N,n}$ of largest modulus approach the points β_j, $j=\pm 1,..., \pm I$. For example, let

$$x(m)=e^{-i\pi m/4}+e^{i\pi m/4}=2\cos(\pi m/4),$$

so that $\beta_{-1}=e^{-i\pi/4}$, $\beta_1=e^{i\pi/4}$, and

$$\Phi_2(z)=z^2-\sqrt{2}\,z+1.$$

In the case when $N \equiv 0 \pmod 4$ formula (1.3) gives

$$\mu_0^{(N)}=2N, \quad \mu_1^{(N)}=\sqrt{2}\,(N-2), \quad \mu_2^{(N)}=-2, \quad \mu_3^{(N)}=-\sqrt{2}\,(N-2),$$

from which we find, using the determinant representation, that

$$\lim_{k\to\infty} \Phi_{4k,3}(z)=\Phi_2(z)\left(z+\frac{\sqrt{2}}{3}\right). \tag{2.7}$$

On the other hand, when $N \equiv 2 \pmod 4$ we get

$$\mu_0^{(N)}=2N+2, \quad \mu_1^{(N)}=\sqrt{2}\,N, \quad \mu_2^{(N)}=0, \quad \mu_3^{(N)}=-\sqrt{2}\,(N-2),$$

from which we deduce that

$$\lim_{k \to \infty} \Phi_{4k+2,3}(z) = \Phi_2(z)\left(z + \frac{\sqrt{2}}{4}\right). \tag{2.8}$$

Thus, the sequence $\{\Phi_{N,3}(z)\}_1^\infty$ does not have a unique limit polynomial.

3. Proofs

Let $\{\mu_k\}_{-\infty}^\infty$ denote the moments of the discrete distribution $d\psi$ defined in (2.2), that is,

$$\mu_k := \int \bar{z}^k \, d\psi = \sum_{j=-I}^{I} |\alpha_j|^2 \, \bar{\beta}_j^k, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{3.1}$$

For $\alpha_0 > 0$, the measure $d\psi$ is supported in $2I+1$ distinct points and so the sequence $\{\mu_k\}_{-\infty}^\infty$ is a positive $(2I+1)$-definite Hermitian sequence. This means that

$$\Delta_n > 0 \text{ for } 0 \leqslant n \leqslant 2I, \text{ and } \Delta_{2I+1} = 0,$$

where

$$\Delta_n := \det(\mu_{i-j})_0^n, \qquad \mu_{-k} = \bar{\mu}_k, \; k = 0, 1, \ldots. \tag{3.2}$$

Thus, the monic orthogonal polynomials $\Phi_n(z)$ with respect to $d\psi$ can be written as (cf. [Sz, p. 288])

$$\Phi_n(z) = \frac{1}{\Delta_{n-1}} \det \begin{pmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & & & \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_{-1} \\ 1 & z & \cdots & z^n \end{pmatrix} \tag{3.3}$$

for $n = 1, 2, \ldots, 2I+1$.

For the moments $\mu_k^{(N)}$ of (1.6) corresponding to the measure $d\psi_N$ in (1.4) we similarly define

$$\Delta_n^{(N)} := \det(\mu_{i-j}^{(N)})_0^n, \tag{3.4}$$

which are positive for all $n \geqslant 0$. The monic orthogonal polynomials $\Phi_{N,n}(z)$ with respect to $d\psi_N$ then have the same representation as in (3.3) with μ_k replaced by $\mu_k^{(N)}$. Moreover, this representation is valid for every $n \geqslant 1$.

From the weak-star convergence (1.10), it follows that

$$\lim_{N\to\infty} \frac{1}{N}\mu_k^{(N)} = \mu_k, \qquad k=0, \pm 1, \pm 2, \ldots.$$

More precisely, we have

LEMMA 3.1. *For each fixed* $k=0, \pm 1, \pm 2, \ldots$,

$$\frac{1}{N}\mu_k^{(N)} = \mu_k + O(1/N) \qquad as \quad N\to\infty. \tag{3.5}$$

As noted in [JTNW] and [P], this result easily follows from (1.3).

Proof of Theorem 2.1. From Lemma 3.1 we have for each fixed $n\geqslant 1$,

$$\frac{1}{N^n}\Delta_{n-1}^{(N)} = \Delta_{n-1} + O(1/N) \qquad \text{as} \quad N\to\infty.$$

Thus, from the determinant representations for $\Phi_{N,n}(z)$ and $\Phi_n(z)$ we get for each $n=1, 2, \ldots, 2I+1$,

$$\Phi_{N,n}(z) = \frac{1}{N^n(\Delta_{n-1}+O(1/N))} \det \begin{pmatrix} \mu_0^{(N)} & \mu_{-1}^{(N)} & \cdots & \mu_{-n}^{(N)} \\ \mu_1^{(N)} & \mu_0^{(N)} & \cdots & \mu_{-n+1}^{(N)} \\ \vdots & & & \\ \mu_{n-1}^{(N)} & \mu_{n-2}^{(N)} & \cdots & \mu_{-1}^{(N)} \\ 1 & z & \cdots & z^n \end{pmatrix}$$

$$= \frac{1}{\Delta_{n-1}+O(1/N)} \left\{ \det \begin{pmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & & & \\ \mu_{n-1} & \mu_{n-2} & \cdots & \mu_{-1} \\ 1 & z & \cdots & z^n \end{pmatrix} + O(1/N) \right\}$$

$$= \Phi_N(z) + O(1/N),$$

where $O(1/N)$ is uniform in z on any compact subset of **C**. ▮

Proof of Corollary 2.3. Since $\Phi_{2I+1}(z) = \prod_{j=-I}^{I}(z-\beta_j)$, it follows from (2.4) that

$$|\Phi_{N,2I+1}(\beta_j)| = O(1/N), \qquad -I\leqslant j\leqslant I. \tag{3.6}$$

As previously remarked, $\beta_{N,j} \to \beta_j$ as $N \to \infty$, for $-I \leqslant j \leqslant I$. Thus, for N large, we have

$$|\beta_j - \beta_{N,j}| = \left| \frac{\Phi_{N,2I+1}(\beta_j)}{\prod_{s \neq j} (\beta_j - \beta_{N,s})} \right| = O(1/N) \qquad \text{as} \quad N \to \infty$$

for $j = -I, ..., I$. ∎

The proof of Theorem 2.4 requires some preliminary lemmas.

Lemma 3.2. *If $\alpha_0 > 0$, then, for $n \geqslant 2I+1$ and any $N \geqslant 1$, the leading coefficients $\kappa_{N,n}$ of the orthogonal polynomials $\phi_{N,n}(z)$ in (1.7) satisfy*

$$\kappa_{N,n} \geqslant 1/\tau > 0, \tag{3.7}$$

where $\tau := 2^{2I+1} \sum_{j=-I}^{I} |\alpha_j|$.

Proof. From the extremal property (1.9) and the definition of W_N in (1.11) it follows that for any $n \geqslant 2I+1$ and $z = e^{i\theta}$ we have

$$\begin{aligned}
\frac{1}{\kappa_{N,n}^2} &= \min_{p_n = z^n + \cdots} \frac{1}{2\pi} \int_{-\pi}^{\pi} |p_n(z)|^2 \, |W_N(z)|^2 \, d\theta \\
&\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| z^{n-2I-1} \prod_{j=-I}^{I} (z - \beta_j) \right|^2 |W_N(z)|^2 \, d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{j=-I}^{I} \alpha_j (z^N - \beta_j^N) \prod_{l \neq j} (z - \beta_l) \right|^2 d\theta \\
&\leqslant 2^{4I+2} \left(\sum_{j=-I}^{I} |\alpha_j| \right)^2 = \tau^2,
\end{aligned}$$

which proves (3.7). ∎

Lemma 3.3. *For all $n \geqslant 2I+1$ and $N = 1, 2, ...$, the reflection coefficients $\Phi_{N,n+1}(0)$ satisfy*

$$|\Phi_{N,n+1}(0)| \leqslant (1 - (x(0)/\tau)^2)^{1/2} < 1,$$

where τ is given in Lemma 3.2.

Proof. We use the fact (cf. [G. p. 7]) that

$$|\Phi_{N,n+1}(0)|^2 = 1 - \kappa_{N,n}^2 / \kappa_{N,n+1}^2. \tag{3.8}$$

Since $\Phi^*_{N,n+1}(z)$ and $W^*_N(z)$ are analytic in $|z| \leqslant 1$, we have from (1.9)

$$\frac{1}{\kappa^2_{N,n+1}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Phi^*_{N,n+1}(z)|^2 \, |W^*_N(z)|^2 \, d\theta$$
$$\geqslant |\Phi^*_{N,n+1}(0) \, W^*_N(0)|^2 = x^2(0),$$

which, together with (3.7), yields

$$\kappa^2_{N,n}/\kappa^2_{N,n+1} \geqslant (x(0)/\tau)^2. \tag{3.9}$$

Hence the lemma follows from (3.8) and (3.9). ∎

LEMMA 3.4. *Let $Q(\not\equiv 0)$ denote any polynomial all of whose zeros lie in $|z| < 1$. If $\lambda \in \mathbf{C}$ satisfies $|\lambda| < 1$, then all the zeros of the polynomial*

$$T(z) := Q^*(z) + z\lambda Q(z) \tag{3.10}$$

lie in $|z| > 1$.

Proof. Suppose to the contrary that $T(z_0) = 0$ and $|z_0| \leqslant 1$. Then from (3.10), we see that $z_0 \neq 0$ and $|Q^*(z_0)| = |z_0 \lambda Q(z_0)|$. Therefore, since $Q^*(z_0) \neq 0$, we have

$$1 = |z_0 \lambda| \, |Q(z_0)/Q^*(z_0)| \leqslant |z_0 \lambda| < |z_0|,$$

where the next to last inequality follows since Q/Q^* is a Blaschke product with all zeros in $|z| < 1$. This contradiction completes the proof. ∎

Proof of Theorem 2.4. Clearly the theorem will be proved if we show that for each fixed $n \geqslant 2I + 1$, every limit polynomial of $\{\Phi_{N,n}\}_{N=1}^{\infty}$ is of the form

$$\Phi_{2I+1} Q_{n-2I-1}, \tag{3.11}$$

where Q_{n-2I-1} is a monic polynomial of degree $n - 2I - 1$ that has all its zeros in the *open* unit disk $|z| < 1$. (Recall the representation of $\Phi_{2I+1}(z)$ given in (2.3).) We proceed to prove this by induction on n.

For $n = 2I + 1$, this assertion follows from Theorem 2.1. Now assume that it is true for $n = m (\geqslant 2I + 1)$ and consider the recurrence formula

$$\Phi^*_{N,m+1}(z) = \Phi^*_{N,m}(z) + z \, \overline{\Phi_{N,m+1}(0)} \, \Phi_{N,m}(z) \tag{3.12}$$

(cf. [Sz, p. 293]). Without loss of generality, we restrict ourselves to a subsequence $\{N_k\}$ such that, as $k \to \infty$,

$$\Phi_{N_k,m+1}(z) \to R(z) \in \mathscr{P}_{m+1}, \quad \text{and} \quad \Phi_{N_k,m}(z) \to S(z) \in \mathscr{P}_m, \tag{3.13}$$

where the convergence is locally uniform in $\mathbf{C}$. (Recall that all the zeros of the $\Phi_{N,n}(z)$'s lie in $|z|<1$ and hence for each fixed n, $\{\Phi_{N,n}\}_{N=1}^{\infty}$ is a normal family in $\mathbf{C}$). Then we have from (3.12)

$$R^*(z) = S^*(z) + z\,\overline{R(0)}\,S(z). \tag{3.14}$$

By the induction hypothesis, we have $S(z) = \Phi_{2I+1}(z)\,Q(z)$, where $Q(z)$ is a monic polynomial of degree $m-2I-1$ having all its zeros in $|z|<1$. Thus (3.14) can be written as

$$\begin{aligned} R^*(z) &= \Phi^*_{2I+1}(z)\,Q^*(z) + z\,\overline{R(0)}\,\Phi_{2I+1}(z)\,Q(z) \\ &= \Phi^*_{2I+1}(z)[Q^*(z) + z\,\overline{R(0)}\,Q(z)], \end{aligned} \tag{3.15}$$

where we have used the fact that $\Phi_{2I+1}(z) = \Phi^*_{2I+1}(z)$ which follows since $\Phi_{2I+1}(z)$ has all its zeros on the unit circle and has real coefficients.

The proof will be complete if we show that $T(z) := Q^*(z) + z\,\overline{R(0)}\,Q(z)$ has all its zeros in $|z|>1$. This follows from Lemma 3.4 since, by Lemma 3.3,

$$|\overline{R(0)}| = \lim_{k\to\infty} |\Phi_{N_k,m+1}(0)| \leqslant (1-(x(0)/\tau)^2)^{1/2} < 1. \quad \blacksquare$$

4. A Related Result

The results of Section 3 apply only in the case when n is fixed; that is, when dealing with Szegő polynomials of fixed degree. Here we present a related result that is valid uniformly in n.

THEOREM 4.1. *For the signal* (1.1), *suppose that* $\alpha_0 > 0$. *Then for every* $n \geqslant 2I+1$, *the reverse orthonormal polynomials* $\phi^*_{N,n}$ *satisfy*

$$\phi^*_{N,n}(r_N\beta_j) = O(1/\sqrt{N}) \qquad \text{as} \quad N\to\infty, \quad -I\leqslant j\leqslant I, \tag{4.1}$$

where $\beta_j := e^{i\omega_j}$, $r_N := 1-1/N$ *and* (4.1) *holds uniformly in n. In case* $\alpha_0 = 0$, *then* (4.1) *holds for any* $n \geqslant 2I$ *and* $j \neq 0$.

We remark that for $n = 2I+1$ it easily follows from Theorem 2.1 that the right-hand side of (4.1) can be replaced by $O(1/N)$.

Before giving the proof of Theorem 4.1, we collect in the following lemma some well known properties of Szegő polynomials (cf. [G, Sz]).

LEMMA 4.2. *For every fixed N and* $|z|<1$, *we have*

(i) $\sum_{k=0}^{n} |\phi_{N,k}(z)|^2 \leqslant |\phi^*_{N,n}(z)|^2/(1-|z|^2)$,

(ii) $\kappa_{N,n}\phi^*_{N,n}(z) = \sum_{k=0}^{n} \overline{\phi_{N,k}(0)}\,\phi_{N,k}(z)$,

(iii) $\lim_{n\to\infty} \phi^*_{N,n}(z) = D_N(0)\sum_{k=0}^{\infty} \overline{\phi_{N,k}(0)}\,\phi_{N,k}(z) = D_N(z)^{-1}$,

(iv) $\sum_{k=n+1}^{\infty} |\phi_{N,k}(0)|^2 = D_N(0)^{-2} - \kappa^2_{N,n}$,

where $D_N(z)$ *denotes the Szegő function for* $d\psi_N$ (*cf.* (1.12)).

Proof of Theorem 4.1. From Lemma 4.2 and the Cauchy–Schwarz inequality we have

$$\begin{aligned}
&|D_N(0)^{-1} D_N(z)^{-1} - \kappa_{N,n}\phi^*_{N,n}(z)| \\
&\quad = \left| \sum_{k=n+1}^{\infty} \overline{\phi_{N,k}(0)}\, \phi_{N,k}(z) \right| \\
&\quad \leqslant \left(\sum_{k=n+1}^{\infty} |\phi_{N,k}(0)|^2 \sum_{k=n+1}^{\infty} |\phi_{N,k}(z)|^2 \right)^{1/2}. \qquad (4.2)
\end{aligned}$$

Furthermore, from (i) and (iii) of Lemma 4.2 it follows that

$$\sum_{k=n+1}^{\infty} |\phi_{N,k}(z)|^2 \leqslant \sum_{k=0}^{\infty} |\phi_{N,k}(z)|^2 \leqslant \frac{|D_N(z)|^{-2}}{1-|z|^2}, \qquad |z|<1.$$

Hence, from (4.2) and (iv) of Lemma 4.2 we get

$$|D_N(0)^{-1} D_N(z)^{-1} - \kappa_{N,n}\phi^*_{N,n}(z)| \leqslant \frac{D_N(0)^{-1}\, |D_N(z)|^{-1}}{\sqrt{1-|z|^2}},$$

and so

$$|\phi^*_{N,n}(z)| \leqslant \frac{D_N(0)^{-1}\, |D_N(z)|^{-1}}{\kappa_{N,n}} \left(1 + \frac{1}{\sqrt{1-|z|^2}}\right), \qquad |z|<1. \qquad (4.3)$$

From the representation (1.12) it follows that

$$D_N(0)^{-1} = \frac{1}{\pm x(0) \prod_{|z_{N,k}| \geqslant 1} (-z_{N,k})} \leqslant \frac{1}{|x(0)|},$$

and from Lemma 3.2 we have $1/\kappa_{N,n} \leqslant \tau$ for all N and all $n \geqslant 2I+1$. Thus from (4.3) we obtain for $|z|<1$

$$\begin{aligned}
|\phi^*_{N,n}(z)| &\leqslant \frac{\tau\, |D_N(z)|^{-1}}{|x(0)|} \left(1 + \frac{1}{\sqrt{1-|z|^2}}\right) \\
&= \frac{\tau}{|x(0)|^2 \prod_{|z_{N,k}| \geqslant 1} |z - z_{N,k}| \prod_{|z_{n,k}|<1} |1 - \bar{z}_{N,k} z|} \\
&\quad \times \left(1 + \frac{1}{\sqrt{1-|z|^2}}\right) \\
&= \frac{\tau}{|x(0)\, W_N(z)|} \prod_{|z_{N,k}|<1} \left| \frac{z - z_{N,k}}{1 - \bar{z}_{N,k} z} \right| \left(1 + \frac{1}{\sqrt{1-|z|^2}}\right) \\
&\leqslant \frac{\tau}{|x(0)\, W_N(z)|} \left(1 + \frac{1}{\sqrt{1-|z|^2}}\right). \qquad (4.4)
\end{aligned}$$

Now if $\alpha_j \neq 0$ and $z = (1 - 1/N)\, e^{i\omega_j} = r_N \beta_j$, we see from (1.11) that $|W_N(z)| \geqslant c_1 N$ for some positive constant c_1 and all N large. Furthermore,

$$\frac{1}{\sqrt{1-|z|^2}} \leqslant \sqrt{N}.$$

Hence, from (4.4) we obtain

$$|\phi^*_{N,n}(r_N \beta_j)| \leqslant \frac{\tau}{|x(0)|\, c_1 N}(1+\sqrt{N}) = O\left(\frac{1}{\sqrt{N}}\right) \qquad \text{as} \quad N \to \infty. \quad \blacksquare$$

Acknowledgments

Some results similar to those in Section 2 appear in the manuscript [JTNW]. The original version of the present paper, which was primarily based upon Theorem 4.1, was independently and simultaneously written. Some of the improvements in this published version were inspired by [JTNW].

References

[G] L. Ya. Geronimus, "Orthogonal Polynomials," Consultants Bureau, New York, 1961.

[JN] W. B. Jones and O. Njåstad, Applications of Szegő polynomials to digital signal processing, *Rocky Mountain J. Math.* **21** (1991), 387–435.

[JS] W. B. Jones and E. B. Saff, Szegő polynomials and frequency analysis, *in* "Approximation Theory" (G. Anastassiou, Ed.), pp. 341–352, Marcel Dekker, New York, 1992.

[JNS] W. B. Jones, O. Njåstad, and E. B. Saff, Szegő polynomials associated with Wiener–Levinson filters, *J. Comput. Appl. Math.* **32** (1990), 387–406.

[JTNW] W. B. Jones, W. J. Thron, O. Njåstad, and H. Waadeland, Szegő polynomials applied to frequency analysis, *J. Comput. Appl. Math.*, to appear.

[L] N. Levinson, The Wiener RMS (root mean square), error criterion in filter design and prediction, *J. Math. and Physics* **25** (1947), 261–278.

[P] K. Pan, Asymptotics for Szegő polynomials associated with Wiener–Levinson filters, submitted for publication.

[Sa] E. B. Saff, Orthogonal polynomials from a complex perspective, *in* "Orthogonal Polynomials: Theory and Practice" (Paul Nevai, Ed.), pp. 363–393, Kluwer, Dordrecht, 1990.

[Sz] G. Szegő, "Orthogonal Polynomials," Colloquium Publication 23, American Mathematical Society Providence, RI, 1939.

[W] Norbert Wiener, "Extrapolation, Interpolation and Smoothing of Stationary Time Series," The Technology Press of the Massachusetts Institute of Technology, and Wiley, New York, 1949.